Invariant Nonholonomic Riemannian Structures on Three-Dimensional Lie Groups

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Introduction

Nonholonomic Riemannian manifold $(M, g, \mathcal{D})$

Model for motion of free particle

- moving in configuration space M
- kinetic energy $L = \frac{1}{2} g(\cdot, \cdot)$
- constrained to move in "admissible directions" $\mathcal{D}$

Invariant structures on Lie groups are of the most interest

Objective

- classify all left-invariant systems on 3D Lie groups
- restrict to unimodular groups

Outline

Outline

1. Invariant nonholonomic Riemannian manifolds
   - Isometries
   - Curvature
2. Unimodular 3D Lie groups
3. Classification of 3D structures
   - Contact structure
   - Case 1
   - Case 2

Invariant nonholonomic Riemannian manifold $(\mathsf{G}, g, \mathcal{D})$

Ingredients

Configuration space G

- n-dim connected Lie group with Lie algebra $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$

Constraint distribution $\mathcal{D} = \{\mathcal{D}_x\}_{x \in \mathsf{G}}$

- left invariant: $\mathcal{D}_x = x\,\mathfrak{d}$, where $\mathfrak{d} \subset \mathfrak{g}$ is an r-dim subspace
- completely nonholonomic: $\mathfrak{d}$ generates $\mathfrak{g}$

Riemannian metric g

- $g_x : T_x\mathsf{G} \times T_x\mathsf{G} \to \mathbb{R}$ is an inner product
- left invariant: $g_x(xU, xV) = g_{\mathbf{1}}(U, V)$ for every $U, V \in \mathfrak{g}$

Orthogonal decomposition $T\mathsf{G} = \mathcal{D} \oplus \mathcal{D}^{\perp}$

- projectors: $\mathscr{P} : T\mathsf{G} \to \mathcal{D}$ and $\mathscr{Q} : T\mathsf{G} \to \mathcal{D}^{\perp}$

Nonholonomic geodesics

Preliminaries

Integral curve of $\mathcal{D}$

- curve γ in G such that $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for every t

Levi-Civita connection $\widetilde\nabla$ of g

- "directional derivative" of one vector field along another

D'Alembert Principle

An integral curve γ of $\mathcal{D}$ is called a nonholonomic geodesic of $(\mathsf{G}, g, \mathcal{D})$ if

$$\widetilde\nabla_{\dot\gamma(t)}\dot\gamma(t) \in \mathcal{D}^{\perp}_{\gamma(t)} \text{ for all } t$$

Equivalently: $\mathscr{P}(\widetilde\nabla_{\dot\gamma(t)}\dot\gamma(t)) = 0$ for every t.

Nonholonomic connection

NH connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$\nabla_X Y = \mathscr{P}(\tilde{\nabla}_X Y), \qquad X, Y \in \Gamma(\mathcal{D})$$

- affine connection
- parallel transport only along integral curves of $\mathcal{D}$
- depends only on $\mathcal{D}$, $g|_{\mathcal{D}}$ and a choice of complement to $\mathcal{D}$

Characterisation of nonholonomic geodesics

integral curve γ of $\mathcal{D}$ is a nonholonomic geodesic $\iff$ $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for every t

Isometries

Isometry between $(\mathsf{G}, g, \mathcal{D})$ and $(\mathsf{G}', g', \mathcal{D}')$

diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}' \qquad \phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp \qquad g|_{\mathcal{D}} = \phi^* g'|_{\mathcal{D}'}$$

Properties of isometries

- preserves NH connection: $\nabla = \phi^* \nabla'$
- 1-to-1 correspondence between NH geodesics of isometric structures

Curvature

- ∇ is not a connection on the vector bundle $\mathcal{D} \to \mathsf{G}$
- hence Riemannian curvature tensor not defined

Schouten curvature tensor $K : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$K(X, Y; Z) = [\nabla_X, \nabla_Y]Z - \nabla_{\mathscr{P}([X,Y])}Z - \mathscr{P}([\mathscr{Q}([X, Y]), Z])$$

- define $\widehat{K}(W, X; Y, Z) = g(K(W, X; Y), Z)$
  - (S1) $\widehat{K}(X, X; Y, Z) = 0$
  - (S2) $\widehat{K}(W, X; Y, Z) + \widehat{K}(X, Y; W, Z) + \widehat{K}(Y, W; X, Z) = 0$
- also define

  $\widehat{R}$:= component of $\widehat{K}$ that is skew-symmetric in second two args

  $\widehat{C} := \widehat{K} - \widehat{R}$
- $\widehat{R}$ behaves like Riemannian curvature tensor

Ricci-like curvatures

Ricci curvature $\mathrm{Ric} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \mathcal{C}^\infty(\mathsf{G})$

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{r} \widehat{R}(X_i, X; Y, X_i)$$

- $(X_i)_{i=1}^r$ is an orthonormal frame for $\mathcal{D}$
- $S := \sum_{i=1}^r \mathrm{Ric}(X_i, X_i)$ is the scalar curvature

Ricci-type tensors $A_{sym}, A_{skew} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \mathcal{C}^\infty(\mathsf{G})$

$$A(X, Y) = \sum_{i=1}^{r} \widehat{C}(X_i, X; Y, X_i)$$

- A_{sym} := symmetric part of A
- A_{skew} := skew-symmetric part of A

Curvature in 3D

Curvature invariants κ, χ_1, χ_2

$$\kappa = \tfrac{1}{2} S \qquad \chi_1 = \sqrt{-\det(g|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{sym})} \qquad \chi_2 = \sqrt{\det(g|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{skew})}$$

- preserved by isometries (i.e., isometric invariants)
- κ, χ_1, χ_2 determine K
- left invariant, hence constant
- for unimodular groups: $\chi_2 = 0$

Outline

The unimodular 3D Lie groups

Bianchi-Behr classification (unimodular algebras)

Lie algebra	Lie group	Name	Class
$\mathbb{R}^3$	$\mathbb{R}^3$	Abelian	Abelian
$\mathfrak{h}_3$	H_3	Heisenberg	Nilpotent
$\mathfrak{se}(1,1)$	$\mathsf{SE}(1,1)$	Semi-Euclidean	Completely solvable
$\mathfrak{se}(2)$	$\widetilde{\mathsf{SE}}(2)$	Euclidean	Solvable
$\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	Special linear	Semisimple
$\mathfrak{su}(2)$	$\mathsf{SU}(2)$	Special unitary	Semisimple

Left-invariant distributions on 3D groups

Killing form

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \mathcal{K}(U, V) = \operatorname{tr}[U, [V, \cdot\,]]$$

- $\mathcal{K}$ is nondegenerate $\iff$ $\mathfrak{g}$ is semisimple

Completely nonholonomic distributions on 3D groups

- no such distributions on $\mathbb{R}^3$

Up to Lie group automorphism:

- exactly one distribution on H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$, $\mathsf{SU}(2)$
- exactly two distributions on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$:

denote	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$	if $\mathcal{K}$ indefinite on $\mathcal{D}$
"	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$	" " definite " "

Outline

Contact structure

Contact form ω on G

We have $\mathcal{D} = \ker \omega$, where $\omega : \mathfrak{X}(\mathsf{G}) \to C^\infty(\mathsf{G})$ is a 1-form such that

$$\omega \wedge d\omega \neq 0$$

- specified up to sign by condition:

$$d\omega(Y_1, Y_2) = \pm 1, \qquad \{Y_1, Y_2\} \text{ o.n. frame for } \mathcal{D}$$

- Reeb vector field $Y_0 \in \mathfrak{X}(\mathsf{G})$:

$$\omega(Y_0) = 1 \qquad \text{and} \qquad d\omega(Y_0, \cdot\,) \equiv 0$$

Two natural cases

(1) $Y_0 \in \mathcal{D}^\perp$ (2) $Y_0 \notin \mathcal{D}^\perp$

A fourth invariant

Extension of $g|_{\mathcal{D}}$ depending on $\mathcal{D}$, $g|_{\mathcal{D}}$

- extend $g|_{\mathcal{D}}$ to a Riemannian metric $\tilde{g}$ such that

$$Y_0 \perp_{\tilde{g}} \mathcal{D} \qquad \text{and} \qquad \tilde{g}(Y_0, Y_0) = 1.$$

- angle θ between Y_0 and $\mathcal{D}^\perp$ is given by

$$\cos\theta = \frac{|\tilde{g}(Y_0, Y_3)|}{\sqrt{\tilde{g}(Y_3, Y_3)}}, \qquad 0 \leq \theta < \tfrac{\pi}{2}, \qquad \mathcal{D}^\perp = \operatorname{span}\{Y_3\}$$

- fourth isometric invariant: $\vartheta := \tan^2\theta \geq 0$

$$Y_0 \in \mathcal{D}^\perp \qquad \Longleftrightarrow \qquad \vartheta = 0$$

Case 1: $\vartheta = 0$

- $\mathcal{D}^{\perp}$ determined by $\mathcal{D}$, $g|_{\mathcal{D}}$
- reduces to a sub-Riemannian structure:

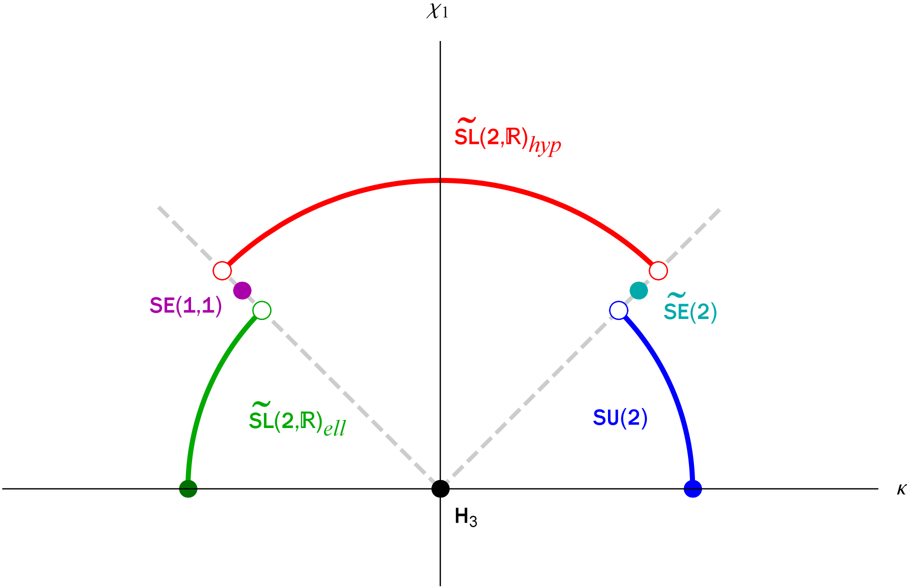
  A. Agrachev and D. Barilari, Sub-Riemannian structures on 3D Lie groups, *J. Dyn. Control Syst.* **18**(2012), 21–44.

Invariants

- $\{\kappa, \chi_1\}$ complete set of invariants (at least for unimodular case)
- can rescale structures so that

$$\kappa = \chi_1 = 0 \qquad \text{or} \qquad \kappa^2 + \chi_1^2 = 1$$

Classification for case 1



Case 2: $\vartheta > 0$

Canonical frame (X_0, X_1, X_2)

$$X_0 = \mathscr{Q}(Y_0) \qquad X_1 = \frac{\mathscr{P}(Y_0)}{\|\mathscr{P}(Y_0)\|} \qquad X_2 \text{ unique unit vector s.t. } d\omega(X_1, X_2) = 1$$

- $\mathcal{D} = \mathrm{span}\{X_1, X_2\}$, $\mathcal{D}^{\perp} = \mathrm{span}\{X_0\}$
- canonical left-invariant frame (up to sign of X_0, X_1) on G

Commutator relations (determine structure uniquely)

$$\begin{cases} [X_1, X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = -c_{21}^1 X_0 + c_{20}^1 X_1 - c_{10}^1 X_2 \\ [X_2, X_1] = X_0 + c_{21}^1 X_1 \end{cases} \qquad c_{10}^1, c_{10}^2, c_{20}^1, c_{21}^1 \in \mathbb{R}, \quad c_{21}^1 > 0$$

Isometries are isomorphisms

Proposition

$(\mathsf{G}, g, \mathcal{D})$ isometric to $(\mathsf{G}', g', \mathcal{D}')$ w.r.t. $\phi : \mathsf{G} \to \mathsf{G}'$ $\implies$ ϕ is a Lie group isomorphism

- hence isometries preserve Killing form $\mathcal{K}$

Three new invariants ϱ_0, ϱ_1, ϱ_2

$$\varrho_i = -\tfrac{1}{2}\mathcal{K}(X_i, X_i), \qquad i = 0, 1, 2$$

- κ, χ_1 expressible i.t.o. ϱ_i's and ϑ
- ϱ_0, ϱ_1, ϱ_2 simpler than κ, χ_1, χ_2 and have more info

Classification

Approach

- rescale frame s.t. $\vartheta = 1$
- split into cases, and read off algebras from commutator relations

Example: case $c_{10}^1 = c_{10}^2 = 0$

$$[X_1, X_0] = 0 \qquad [X_2, X_0] = X_0 + c_{20}^1 X_1 \qquad [X_2, X_1] = X_0 - X_1$$

- implies $\mathcal{K}$ is degenerate (i.e., G not semisimple)

(1) $c_{20}^1 + 1 > 0$	$\Longrightarrow$	compl. solvable	hence on $\mathsf{SE}(1,1)$
(2) $c_{20}^1 + 1 = 0$	$\Longrightarrow$	nilpotent	" " H_3
(3) $c_{20}^1 + 1 < 0$	$\Longrightarrow$	solvable	" " $\widetilde{\mathsf{SE}}(2)$

- for $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$: c_{20}^1 is a parameter (i.e., family of structures)

Results (solvable groups)

H_3 :
$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = X_0 - X_1 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = 0 \end{cases}$$

$\mathsf{SE}(1,1)$:
$$\begin{cases} [X_1, X_0] = \sqrt{\alpha_1\alpha_2}\, X_1 - \alpha_1 X_2 \\ [X_2, X_0] = X_0 - (1 - \alpha_2) X_1 - \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\alpha_1 \\ \varrho_1 = -\alpha_2 \\ \varrho_2 = -\alpha_2 \end{cases}$$
$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

$\widetilde{\mathsf{SE}}(2)$:
$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 + \alpha_1 X_2 \\ [X_2, X_0] = X_0 - (1 + \alpha_2) X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1 \\ \varrho_1 = \alpha_2 \\ \varrho_2 = \alpha_2 \end{cases}$$
$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

Results (semisimple groups)

$\mathsf{SU}(2)$:
$$\begin{cases} [X_1, X_0] = -\alpha X_0 - \beta_1 X_2 \\ [X_2, X_0] = X_0 - \beta_2 X_1 + \alpha X_2 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -(\alpha^2 + \beta_1\beta_2) \\ \varrho_1 = -\beta_1 \\ \varrho_2 = \beta_2 - 1 \end{cases}$$
$$(\alpha \geq 0,\ \beta_1, \beta_2 \neq 0) \qquad (\varrho_0 > 0,\ \varrho_1(\varrho_0 - \varrho_1) < 0)$$

$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$:
$$\begin{cases} [X_1, X_0] = -\alpha X_1 - \beta X_2 \\ [X_2, X_0] = X_0 - \gamma X_1 + \alpha X_2 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -(\alpha^2 + \beta\gamma) \\ \varrho_1 = -\beta \\ \varrho_2 = \gamma - 1 \end{cases}$$
$$(\alpha \geq 0,\ \beta \neq 0,\ \gamma \in \mathbb{R}) \qquad (\varrho_0 \leq 0,\ \varrho_1(\varrho_0 - \varrho_1) < 0)$$

$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$:
$$\begin{cases} [X_1, X_0] = -\alpha X_1 - \gamma_1 X_2 \\ [X_2, X_0] = X_0 - \gamma_2 X_1 + \alpha X_2 \\ [X_2, X_1] = X_0 - X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -(\alpha^2 + \gamma_1\gamma_2) \\ \varrho_1 = -\gamma_1 \\ \varrho_2 = \gamma_2 - 1 \end{cases}$$
$$(\alpha \geq 0,\ \gamma_1, \gamma_2 \in \mathbb{R}) \qquad (\varrho_1(\varrho_0 - \varrho_1) \geq 0,\ \varrho_0 \neq \varrho_1)$$

Remarks

- $\{\vartheta, \varrho_0, \varrho_1, \varrho_2\}$ form a complete set of invariants
- (again, only for unimodular case)

Structures on non-unimodular groups

On a fixed 3D non-unimodular Lie group (except for $G_{3.5}^1$), there exist at most two non-isometric structures with the same invariants ϑ, ϱ_0, ϱ_1, ϱ_2

- exception $G_{3.5}^1$: infinitely many ($\varrho_0 = \varrho_1 = \varrho_2 = 0$)
- use κ, χ_1 or χ_2 to form complete set of invariants

Conclusion

Cartan connections

∇ is called a Cartan connection if NH geodesics are of the form

$$\gamma(t) = x_0 \exp(tU_0), \qquad x_0 \in \mathsf{G}, \; U_0 \in \mathfrak{d}$$

- characterisation:

$$\nabla_X Y = \tfrac{1}{2}\mathscr{P}([X, Y]) \text{ for all left-invariant } X, \; Y$$

- 3D unimodular structures: Cartan connection $\iff$ $\vartheta = 0$

Shortest vs straightest curves

- NH geodesics are "straightest" curves (zero geodesic curvature)
- SR geodesics are "shortest" curves (local length minimizers)
- when do we have

$$\{\text{NH geodesics}\} \subset \{\text{SR geodesics}\}?$$